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AUTHOR(S):

Han, Huhe; Nishimura, Takashi

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Wulff shapes and their duals

Huhe Han*

Graduate School of Environment and Information Sciences,
Yokohama National University

Takashi Nishimura†

Research Institute of Environment and Information Sciences,
Yokohama National University,

1 Introduction

The Wulff construction is well-known as a geometric model of an equilibrium crystal defined as follows. Let n be a positive integer. Given a continuous function $\gamma : S^n \rightarrow \mathbb{R}_+$ where $S^n \subset \mathbb{R}^{n+1}$ is the unit sphere and \mathbb{R}_+ is the set consisting of positive real numbers, the *Wulff shape associated with γ* , denoted by \mathcal{W}_γ , is the following intersection (see Figure 1)

$$\mathcal{W}_\gamma = \bigcap_{\theta \in S^n} \Gamma_{\gamma, \theta}.$$

Here, $\Gamma_{\gamma, \theta}$ is the following half-space:

$$\Gamma_{\gamma, \theta} = \{x \in \mathbb{R}^{n+1} \mid x \cdot \theta \leq \gamma(\theta)\}.$$

By Wulff construction, we know that Wulff shape is a compact, convex and contains the the origin of

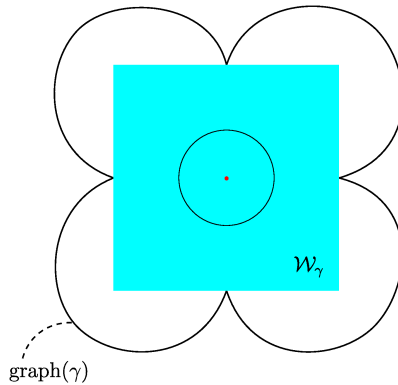


Figure 1: A Wulff shape \mathcal{W}_γ .

\mathbb{R}^{n+1} as an interior point. Conversely, it is well-known that any convex body W contains the origin as

*han-huhe-bx@ynu.jp

†nishimura-takashi-yx@ynu.jp

an interior point is a Wulff shape given by appropriate support function, namely, there is a continuous function $\gamma : S^n \rightarrow \mathbb{R}_+$ such that $\mathcal{W}_\gamma = W$. For details on Wulff shapes, see for instance [1, 6, 13, 14].

For a continuous function $\gamma : S^n \rightarrow \mathbb{R}_+$, set

$$\text{graph}(\gamma) = \{(\theta, \gamma(\theta)) \in \mathbb{R}^{n+1} - \{0\} \mid \theta \in S^n\},$$

where $(\theta, \gamma(\theta))$ is the polar plot expression for a point of $\mathbb{R}^{n+1} - \{0\}$. The mapping $\text{inv} : \mathbb{R}^{n+1} - \{0\} \rightarrow \mathbb{R}^{n+1} - \{0\}$, defined as follows, is called the *inversion* with respect to the origin of \mathbb{R}^{n+1} .

$$\text{inv}(\theta, r) = \left(-\theta, \frac{1}{r}\right).$$

Let Γ_γ be the boundary of the convex hull of $\text{inv}(\text{graph}(\gamma))$.

Definition 1 ([12, 10])

Let $\gamma : S^n \rightarrow \mathbb{R}_+$ be a continuous function. If the equality $\Gamma_\gamma = \text{inv}(\text{graph}(\gamma))$ is satisfied, then γ is called a *convex integrand*.

The notion of convex integrand was firstly introduced by J. Taylor in [12] and it plays a key role for studying Wulff shapes (for details on convex integrands, see for instance [4, 7, 12]).

Definition 2 ([10])

Let $\gamma : S^n \rightarrow \mathbb{R}_+$ be a continuous function. The convex hull of $\text{inv}(\text{graph}(\gamma))$ is called *dual Wulff* of \mathcal{W}_γ , denoted by $D\mathcal{W}_\gamma$.

The main topic of this paper is the relations between Wulff shapes and its duals.

2 Properties and some known results

Before proceeding further, we first introduce an equivalent definition of Wulff shape, given in [10].

(1) $Id : \mathbb{R}^{n+1} \rightarrow \mathbb{R}^{n+1} \times \{1\}$.

Let $Id : \mathbb{R}^{n+1} \rightarrow \mathbb{R}^{n+1} \times \{1\}$ be the map defined by $Id(x) = (x, 1)$.

(2) $\alpha_N : S_{N,+}^{n+1} \rightarrow \mathbb{R}^{n+1} \times \{1\}$.

Denote the point $(0, \dots, 0, 1) \in \mathbb{R}^{n+2}$ by N . The set $S^{n+1} - H(-N)$ is denoted by $S_{N,+}^{n+1}$. Let $\alpha_N : S_{N,+}^{n+1} \rightarrow \mathbb{R}^{n+1} \times \{1\}$ be the central projection relative to N , namely, α_N is defined as follows for any $P = (P_1, \dots, P_{n+1}, P_{n+2}) \in S_{N,+}^{n+1}$:

$$\alpha_N(P_1, \dots, P_{n+1}, P_{n+2}) = \left(\frac{P_1}{P_{n+2}}, \dots, \frac{P_{n+1}}{P_{n+2}}, 1\right).$$

(3) $\Psi_N : S^{n+1} - \{\pm N\} \rightarrow S_{N,+}^{n+1}$.

Next, we consider the mapping $\Psi_N : S^{n+1} - \{\pm N\} \rightarrow S_{N,+}^{n+1}$, defined by

$$\Psi_N(\tilde{P}) = \frac{1}{\sqrt{1 - (N \cdot \tilde{P})^2}}(N - (N \cdot \tilde{P})\tilde{P}).$$

The mapping Ψ_N was introduced in [9], has the following intriguing properties:

1. For any $\tilde{P} \in S^{n+1} - \{\pm N\}$, the equality $\tilde{P} \cdot \Psi_N(\tilde{P}) = 0$ holds,
2. for any $\tilde{P} \in S^{n+1} - \{\pm N\}$, the property $\Psi_N(\tilde{P}) \in \mathbb{R}N + \mathbb{R}\tilde{P}$ holds,
3. for any $\tilde{P} \in S^{n+1} - \{\pm N\}$, the property $N \cdot \Psi_N(\tilde{P}) > 0$ holds,
4. the restriction $\Psi_N|_{S_{N,+}^{n+1} - \{N\}} : S_{N,+}^{n+1} - \{N\} \rightarrow S_{N,+}^{n+1} - \{N\}$ is a C^∞ diffeomorphism.

(4) Spherical polar transform.

For any point $\tilde{P} \in S^{n+1}$, let $H(\tilde{P})$ be the closed hemisphere centered at \tilde{P} , namely,

$$H(\tilde{P}) = \{\tilde{Q} \in S^{n+1} | \tilde{P} \cdot \tilde{Q} \geq 0\},$$

where the dot in the center stands for the scalar product of two vectors $\tilde{P}, \tilde{Q} \in \mathbb{R}^{n+2}$. For any non-empty subset $\tilde{W} \subset S^{n+1}$, the *spherical polar set* of \tilde{W} , denoted by \tilde{W}° , is defined as follows:

$$\tilde{W}^\circ = \bigcap_{\tilde{P} \in \tilde{W}} H(\tilde{P}).$$

for details on spherical polar set, see for instance [3, 10]

Proposition 1 ([10])

Let $\gamma : S^n \rightarrow \mathbb{R}_+$ be a continuous function. Let $\text{graph}(\gamma) = \{(\theta, \gamma(\theta)) \in \mathbb{R}^{n+1} - \{0\} \mid \theta \in S^n\}$, where $(\theta, \gamma(\theta))$ is the polar plot expression for a point of $\mathbb{R}^{n+1} - \{0\}$. Then, \mathcal{W}_γ is characterized as follows:

$$\mathcal{W}_\gamma = Id^{-1} \circ \alpha_N \left((\Psi_N \circ \alpha_N^{-1} \circ Id(\text{graph}(\gamma)))^\circ \right).$$

For any Wulff shape \mathcal{W}_γ , by Proposition 1, the dual Wulff shape \mathcal{DW}_γ can be characterized as follows:

Proposition 2 ([10])

For any Wulff shape \mathcal{W}_γ , the following holds:

$$\mathcal{DW}_\gamma = Id^{-1} \circ \alpha_N \left((\alpha_N^{-1} \circ Id(\mathcal{W}_\gamma))^\circ \right).$$

In general case, for given Wulff shape W , there exist uncountable many support function γ construct W (see Figure 2). Then, it is natural arise that "When does given Wulff shape has only one support function?". In [4], it is shown that Wulff shape W is strictly convex if and only if its convex integrand γ is of class C^1 . By this result, the following theorem is not difficult to prove.

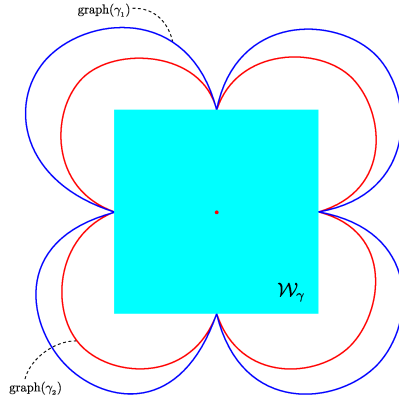


Figure 2: A Wulff shape \mathcal{W}_γ .

Theorem 1 ([6])

Let $\gamma : S^n \rightarrow \mathbb{R}_+$ be a continuous function and let \mathcal{W}_γ be the Wulff shape associated with γ . Suppose that the boundary of \mathcal{W}_γ is a C^1 submanifold. Then, γ must be the convex integrand of \mathcal{W}_γ .

For Wulff shapes and their dual Wulff shapes, we have the following relations.

Proposition 3 ([4])

A Wulff shape in \mathbb{R}^{n+1} is strictly convex if and only if the boundary of its dual Wulff shape is C^1 diffeomorphic to S^n .

Proposition 4 ([4])

A Wulff shape in \mathbb{R}^{n+1} is strictly convex and its boundary is C^1 diffeomorphic to S^n if and only if its dual Wulff shape is strictly convex and the boundary of it is C^1 diffeomorphic to S^n .

Proposition 5 ([10])

A Wulff shape in \mathbb{R}^{n+1} is a polytope if and only if its dual Wulff shape is a polytope.

Definition 3 ([2])

Let γ_1, γ_2 be convex integrands. Define γ_{\max} and γ_{\min} as natural way.

$$\gamma_{\max} : S^n \rightarrow \mathbb{R}_+, \gamma_{\max}(\theta) = \max\{\gamma_1(\theta), \gamma_2(\theta)\}.$$

$$\gamma_{\min} : S^n \rightarrow \mathbb{R}_+, \gamma_{\min}(\theta) = \min\{\gamma_1(\theta), \gamma_2(\theta)\}.$$

Proposition 6 ([2])

Let $\mathcal{W}_{\gamma_1}, \mathcal{W}_{\gamma_2}$ be dual Wulff shapes. Then $\mathcal{W}_{\gamma_{\max}}$ is the dual Wulff shape of $\mathcal{W}_{\gamma_{\min}}$.

3 Self-dual Wulff shapes

Definition 4 ([5])

Let W be a Wulff shape. If Wulff shape W and its dual Wulff shape are same convex body, then W is said to be *self-dual* Wulff shape.

By Proposition 1, we have the following.

Corollary 1

Let $\gamma : S^n \rightarrow \mathbb{R}_+$ be a continuous function. Then the following are equivalent.

1. $\mathcal{W}_\gamma = \mathcal{DW}_\gamma$.
2. $\mathcal{W}_\gamma = Id^{-1} \circ \alpha_N \left((\alpha_N^{-1} \circ Id(\mathcal{W}_\gamma))^\circ \right)$.
3. \mathcal{W}_γ is exactly the convex hull of $\text{inv}(\text{graph}(\gamma))$.

Moreover, self-dual Wulff shape can be characterized as follows.

Definition 5 ([3])

1. A subset \widetilde{W} of S^{n+1} is said to be *hemispherical* if there exists a point $\widetilde{P} \in S^{n+1}$ such that $\widetilde{W} \cap H(\widetilde{P}) = \emptyset$.
2. A hemispherical subset $\widetilde{W} \subset S^{n+1}$ is said to be *spherical convex* if for any $\widetilde{P}, \widetilde{Q} \in \widetilde{W}$ the following arc $\widetilde{P}\widetilde{Q}$ is contained in \widetilde{W} :

$$\widetilde{P}\widetilde{Q} = \left\{ \frac{(1-t)\widetilde{P} + t\widetilde{Q}}{\|(1-t)\widetilde{P} + t\widetilde{Q}\|} \mid t \in [0, 1] \right\}.$$

3. A hemispherical subset \widetilde{W} is called a *spherical convex body* if it is closed, spherical convex and has an interior point. A hemisphere $H(\widetilde{P})$ is said to *support* a spherical convex body \widetilde{W} if both $\widetilde{W} \subset H(\widetilde{P})$ and $\partial\widetilde{W} \cap \partial H(\widetilde{P}) \neq \emptyset$ hold.

Definition 6 ([8])

1. For any two $\tilde{P}, \tilde{Q} \in S^{n+1}$ ($\tilde{P} \neq \pm \tilde{Q}$), the intersection $H(\tilde{P}) \cap H(\tilde{Q})$ is called a *lune* of S^{n+1} .
2. The *thickness of the lune* $H(\tilde{P}) \cap H(\tilde{Q})$, denoted by $\Delta(H(\tilde{P}) \cap H(\tilde{Q}))$, is the real number $\pi - |\tilde{P}\tilde{Q}|$, where $|\tilde{P}\tilde{Q}|$ stands for the length of the arc $\tilde{P}\tilde{Q}$.
3. For a spherical convex body \tilde{W} and a hemisphere $H(\tilde{P})$ supporting \tilde{W} , the *width of \tilde{W} determined by $H(\tilde{P})$* , denoted by $\text{width}_{H(\tilde{P})}\tilde{W}$, is the minimum of the following set:

$$\left\{ \Delta(H(\tilde{P}) \cap H(\tilde{Q})) \mid \tilde{W} \subset H(\tilde{P}) \cap H(\tilde{Q}), H(\tilde{Q}) \text{ supports } \tilde{W} \right\}.$$

4. For any $\rho \in \mathbb{R}_+$ less than π , a spherical convex body $\tilde{W} \subset S^{n+1}$ is said to be of *constant width ρ* if $\text{width}_{H(\tilde{P})}\tilde{W} = \rho$ for any $H(\tilde{P})$ supporting \tilde{W} .

Theorem 2 ([5])

Let $\gamma : S^n \rightarrow \mathbb{R}_+$ be a continuous function. Then, the Wulff shape W_γ is self-dual if and only if the spherical convex body $\tilde{W}_\gamma = \alpha_N^{-1} \circ \text{Id}(W_\gamma)$ is of constant width $\pi/2$.

Definition 7 ([8])

Let \tilde{W} be a spherical convex body of S^{n+1} .

1. Thickness $\Delta(\tilde{W})$ of $\tilde{W} \subset S^{n+1}$ defined as follows:

$$\Delta(\tilde{W}) = \inf \{ \text{width}_K(\tilde{W}); K \text{ is a supporting hemisphere of } \tilde{W} \}.$$

2. $\tilde{W} \subset S^{n+1}$ is said to be *reduced* if $\Delta(\tilde{Y}) < \Delta(\tilde{W})$ for every convex body $\tilde{Y} \subset \tilde{W}$ different from \tilde{W} .

Theorem 3 ([8])

Every smooth reduced body on S^n is of constant width.

In the case of Wulff shapes, the following seems to be open.

Definition 8 ([8])

Let $\tilde{W} \subset S^{n+1}$ be a spherical convex body. Then, the following number is called the *diameter* of \tilde{W} and is denoted by $\text{diam}(\tilde{W})$:

$$\max \left\{ |\tilde{P}\tilde{Q}| \mid \tilde{P}, \tilde{Q} \in \tilde{W} \right\}.$$

Question: Let W be a Wulff shape. Are the following equivalent?

1. Wulff shape W is self-dual.
2. Spherical convex body $\tilde{W}_\gamma = \alpha_N^{-1} \circ \text{Id}(W_\gamma)$ is reduced and $\text{diam}(\tilde{W}) = \pi/2$.

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